

FULL CUNTZ-KRIEGER DILATIONS VIA NON-COMMUTATIVE BOUNDARIES

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*We dedicate this article to the memory of Itzik Martziano,
who was an operator theorist, a brilliant colleague and a friend.*

ABSTRACT. We apply Arveson's non-commutative boundary theory to dilate every Toeplitz-Cuntz-Krieger family of a directed graph G to a full Cuntz-Krieger family for G . We do this by describing all representations of the Toeplitz algebra $\mathcal{T}(G)$ that have unique extension when restricted to the tensor algebra $\mathcal{T}_+(G)$. This yields an alternative proof to a result of Katsoulis and Kribs that the C^* -envelope of $\mathcal{T}_+(G)$ is the Cuntz-Krieger algebra $\mathcal{O}(G)$.

We then generalize our dilation result further, to the context of colored directed graphs, by investigating free products of the operator algebras. This relies on a result of independent interest on the complete injectivity of free products of operator algebras amalgamated over a common C^* -algebra.

1. INTRODUCTION

Perhaps the simplest dilation result in operator theory is the dilation of an isometry to a unitary. If $V \in B(H)$ is an isometry and $\Delta := I_H - VV^*$, we may define a unitary U on $K := H \oplus H$ via

$$U := \begin{bmatrix} V & \Delta \\ 0 & V^* \end{bmatrix}$$

such that for any polynomial in a single variable $p \in \mathbb{C}[x]$ we have $p(V) = P_H p(U)|_H$ where P_H is the orthogonal projection onto the first summand of $K = H \oplus H$. One of our goals in this paper is to generalize this dilation result to the free multivariable setting in the context of families of operators arising from directed graphs.

A *directed graph* G is a quadruple (V, E, s, r) consisting of a set V of vertices, a set E of edges and two maps $s, r : E \rightarrow V$, called the

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source map and range map, respectively. If $v = s(e)$ and $w = r(e)$ we say that v emits e and w receives it. In this paper, we will assume that directed graphs are countable, meaning that both the sets V and E are countable. A directed graph is said to be *row-finite* if every vertex receives at most finitely many edges, and is *sourceless* if every vertex receives at least one edge.

The tensor algebra $\mathcal{T}_+(G)$ and the C^* -algebras $\mathcal{T}(G)$ and $\mathcal{O}(G)$ associated to a directed graph G have been studied by many authors. For instance [24, 29, 36, 30] and [12] to name but a few. We recommend [36] and the references therein for additional details.

For a directed graph $G = (V, E, s, r)$ a *Toeplitz-Cuntz-Krieger G -family* (P, S) is a set of mutually orthogonal projections $P := \{P_v : v \in V\}$ and a set of partial isometries $S := \{S_e : e \in E\}$ which satisfy the Toeplitz-Cuntz-Krieger relations:

- (I) $S_e^* S_e = P_{s(e)}$ for every $e \in E$, and
- (TCK) $\sum_{e \in F} S_e S_e^* \leq P_v$ for every finite subset $F \subseteq r^{-1}(v)$.

We say that (P, S) is a *Cuntz-Krieger G -family* if, in addition, we have

- (CK) $\sum_{r(e)=v} S_e S_e^* = P_v$ for every $v \in V$ with $0 < |r^{-1}(v)| < \infty$.

The universal C^* -algebra $\mathcal{T}(G)$ generated by Toeplitz-Cuntz-Krieger G -families is called the *Toeplitz-Cuntz-Krieger algebra* of the graph G , and the universal C^* -algebra $\mathcal{O}(G)$ generated by Cuntz-Krieger families is called the *Cuntz-Krieger algebra* of the graph G . The *tensor algebra* $\mathcal{T}_+(G)$ is then just the norm-closed operator algebra generated by a universal Toeplitz-Cuntz-Krieger family, and is a subalgebra of $\mathcal{T}(G)$.

Due to their universal properties, $*$ -representations of $\mathcal{T}(G)$ are in bijection with TCK G -families, and $*$ -representations of $\mathcal{O}(G)$ are in bijection with CK G -families, and we will often pass freely between these two points of view.

Toeplitz-Cuntz-Krieger families and Cuntz-Krieger families are easily seen to generalize the notions of an isometry and unitary respectively, by taking the graph with a single loop and a single vertex.

In our context of dilation of an isometry to a unitary, Popescu [35, Proposition 2.6] proves that for a countable set F and a row-isometry $V = (V_i)_{i \in F}$ on a space H there is a dilation to a row-unitary. In other words, this means that for any family of isometries $V_i : H \rightarrow H$ such that $\text{SOT-}\sum_{i \in F} V_i V_i^* \leq I_H$ there is a Hilbert space K containing H , and isometries $U = (U_i)_{i \in F}$ on K such that $\text{SOT-}\sum_{i \in F} U_i U_i^* = I_K$, and for any polynomial $p \in \mathbb{C}\langle x_i \rangle_{i \in F}$ in non-commuting variables, we have $p(V) = P_H p(U)|_H$ where P_H is the projection from K to H . In terms of graphs, this means that for a graph with a single vertex and $|F|$

loops, dilation of a TCK family to a CK family is possible, with the extra SOT-convergence $\text{SOT-}\sum_{i \in F} U_i U_i^* = I_K$ when F is infinite.

On the other hand, from [37, Theorem 5.4] we see that if G is row-finite and sourceless, then any TCK family has a CK dilation. More precisely for a row-finite sourceless graph $G = (V, E, s, r)$, if (P, S) is a TCK family on H , then there exists a CK family (Q, T) on a larger space H such that for any polynomial $p \in \mathbb{C}\langle V, E \rangle$ in non-commuting variables we have $p(P, S) = P_H p(Q, T)|_H$.

In order to put both of these results in the same context, we make the following definition.

Definition 1.1. *Let (P, S) be a Cuntz-Krieger family for a countable directed graph G . We say that (P, S) is a full Cuntz-Krieger family if (CKF) $\text{SOT-}\sum_{r(e)=v} S_e S_e^* = P_v$, for every $v \in V$ with $r^{-1}(v) \neq \emptyset$.*

We will show that in dilation theoretic terms, full CK families are the proper generalization of the notion of a unitary operator. More precisely, we will show that every TCK family has a full CK dilation, and that no non-trivial TCK dilations for full CK families are possible (See Corollary 3.7).

We obtain our results by appealing to the non-commutative boundary theory introduced by Arveson [4, 5]. The notions of the C^* -envelope, also known as the non-commutative Shilov boundary, and the more delicate non-commutative Choquet boundary are very useful in operator algebras. A good instance of this appears in the recent work of Katsoulis on the Hao-Ng isomorphism problem [23], where non-self-adjoint algebras and their C^* -envelopes play a prominent role.

Let \mathcal{A} be an operator algebra. We say that the pair (ι, \mathcal{B}) is a C^* -cover for a (not necessarily unital) operator algebra \mathcal{A} , if $\iota : \mathcal{A} \rightarrow \mathcal{B}$ is a completely isometric homomorphism and $C^*(\iota(\mathcal{A})) = \mathcal{B}$. We will often identify \mathcal{A} with its image $\iota(\mathcal{A})$ under a given C^* -cover (ι, \mathcal{B}) for \mathcal{A} . We will call a linear map $\rho : \mathcal{A} \rightarrow B(K)$ a *representation* of \mathcal{A} if it is a completely contractive homomorphism.

By [7, Proposition 4.3.5] there is a unique, smallest C^* -cover for \mathcal{A} . This C^* -cover (\mathcal{B}, ι) is called the C^* -envelope of \mathcal{A} and it satisfies the following universal property: given any other C^* -cover (\mathcal{B}', ι') for \mathcal{A} , there exists a (necessarily unique and surjective) $*$ -homomorphism $\pi : \mathcal{B}' \rightarrow \mathcal{B}$, such that $\pi \circ \iota' = \iota$.

Characterizing the C^* -envelope of various operator structures was of use and intrigue to many authors, as can be seen for instance in [22, 13, 27]. In [25], Katsoulis and Kribs improve on the work in [32] and [19, Theorem 5.3] and show that the C^* -envelope of a tensor algebra associated to a general C^* -correspondence, is the Cuntz-Pimsner-Katsura

algebra of the C^* -correspondence. In particular, the C^* -envelope of the tensor algebra $\mathcal{T}_+(G)$ is the Cuntz-Krieger algebra $\mathcal{O}(G)$ (this was also shown directly in [26]). We will provide an alternative proof for this fact on graph algebras in Theorem 3.9.

Suppose \mathcal{A} is a unital operator algebra generating a C^* -algebra \mathcal{B} . We say that a unital representation $\rho : \mathcal{A} \rightarrow B(H)$ has the *unique extension property* if the only unital completely positive extension to \mathcal{B} is a $*$ -representation.

When \mathcal{A} generates a C^* -algebra \mathcal{B} , Arveson defined boundary representations to be those irreducible $*$ -representation of \mathcal{B} whose restriction to \mathcal{A} has the unique extension property. The collection of all these representations generalizes the classical notion of Choquet boundary for uniform algebras, and is therefore sometimes called the *non-commutative Choquet boundary* for \mathcal{A} .

We say that \mathcal{A} has the unique extension property in \mathcal{B} if for any unital faithful $*$ -representation $\pi : \mathcal{B} \rightarrow B(H)$ we have that $\pi|_{\mathcal{A}}$ has the unique extension property. The unique extension property of operator algebras was used by Kakariadis in [21] to compute C^* -envelopes of various operator algebras. When \mathcal{A} is also separable, the unique extension property of \mathcal{A} inside $C_e^*(\mathcal{A})$ is equivalent to the notion of hyperrigidity, introduced by Arveson in [6]. Hyperrigidity and Arveson's hyperrigidity conjecture have been of interest to several authors recently. For instance, in [15], Davidson and Kennedy verify Arveson's hyperrigidity conjecture for commutative C^* -envelopes, and in the context of the Arveson-Douglas conjecture, Kennedy and Shalit show in [28] that the essential normality of a d -tuple of operators satisfying homogeneous polynomial constraints is equivalent to the hyperrigidity of the d -tuple.

One of our main results is the classification of those $*$ -representation of $\mathcal{T}(G)$ that have the unique extension property when restricted to $\mathcal{T}_+(G)$. They turn out to coincide with those $*$ -representations that are associated with full Cuntz-Krieger families (see Theorem 3.5). This allows us to improve upon several known results and show that any TCK family dilates to a full CK family in the sense described above. Further applications of this result allows us to give a bijective correspondence between irreducible $*$ -representations of $\mathcal{T}(G)$ that are not boundary, and “gap” TCK families of the graph G (see Corollary 3.6), and a characterization of the unique extension property of $\mathcal{T}_+(G)$ in terms of the graph G (see Theorem 3.9).

Trying to leverage our results to free products, we discuss some of the general theory of free products of operator algebras, and prove the existence of joint unital completely positive extension theorem for

free products of operator algebras amalgamated over any common C^* -algebra (see Theorem 4.1). Complete injectivity of amalgamated free products of C^* -algebras was shown by Armstrong, Dykema, Exel and Li [2], and we are able to use our joint extension result to generalize this to free products of operator algebras amalgamated over any common C^* -subalgebra (see Proposition 4.3).

Using complete injectivity, and a stronger joint extension result due to Boca [10], we characterize representations with the unique extension property on amalgamated free products (see Proposition 4.4), and apply our results to free products of graph operator algebras. Free products of graph operator algebras have been investigated by Ara and Goodearl in [1] as C^* -algebras associated to separated graphs, and by Duncan [17] as operator algebras associated to edge-colored directed graphs. We combine our results to prove that a full-CK dilation exists for any TCK family of a colored directed graph (See Corollary 5.2), and to show that the free product of Cuntz-Krieger algebras is a C^* -cover for the free product of tensor graph algebras, which is the C^* -envelope when all graphs involved are row-finite (see Theorem 5.4).

This paper has five sections including this introduction. In Section 2 we discuss some preliminary material on non-commutative boundary theory, especially in the non-unital context. In Section 3 we describe a Wold decomposition for Toeplitz-Cuntz-Krieger families, and characterize representations whose restriction to the tensor algebra has the unique extension property. We use this to obtain our main dilation result and compute the C^* -envelope of graph tensor algebras. In Section 4 we prove a joint extension theorem for free products of operator algebras amalgamated over a common C^* -algebra, and show that in cases with conditional expectations, we have stability of the unique extension property under free products. Finally, in Section 5, we apply the results of Sections 3 and 4 to obtain a free product generalization, providing a free/colored version of our dilation result, and get our C^* -cover results for free products of graph tensor algebras.

2. PRELIMINARIES

2.1. C^* -envelopes, boundary representations and the unique extension property. Operator algebras can be given an axiomatic definition, as was shown in [9]. This means that there is an intrinsic operator structure to these objects that is preserved by any completely isometric homomorphism. We will survey the theory of non-commutative boundaries for operator algebras, and we refer the reader to [4, 5, 3, 7] for a more in-depth treatment of the theory.

For an operator algebra \mathcal{A} generating a C^* -algebra \mathcal{B} , an ideal \mathcal{J} of \mathcal{B} is called a *boundary ideal* for \mathcal{A} if the quotient map $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{J}$ is completely isometric on \mathcal{A} . The largest boundary ideal $\mathcal{J}_S(\mathcal{A})$ of \mathcal{B} is called *the Shilov ideal* of \mathcal{A} in \mathcal{B} , and its importance in our context is that it gives a way to compute the C^* -envelope. Namely, the C^* -envelope of \mathcal{A} is always isomorphic to $\mathcal{B}/\mathcal{J}_S(\mathcal{A})$.

When \mathcal{A} is unital and $\pi : \mathcal{B} \rightarrow B(H)$ is a unital $*$ -representation such that $\pi|_{\mathcal{A}}$ has the unique extension property, every boundary ideal of \mathcal{A} in \mathcal{B} is contained in $\text{Ker}\pi$. The boundary theorem of Davidson and Kennedy [14] then describes the Shilov ideal as the intersection of all kernels of boundary representations, providing another way to compute the C^* -envelope, via the non-commutative Choquet boundary.

For a (not-necessarily-unital) operator algebra \mathcal{A} and a representation $\varphi : \mathcal{A} \rightarrow B(H)$, a representation $\psi : \mathcal{A} \rightarrow B(K)$ is said to *dilate* φ if there is an isometry $V : H \rightarrow K$ such that for all $a \in \mathcal{A}$ we have $\varphi(a) = V^*\psi(a)V$. Since V is an isometry, we can identify $H \cong V(H)$ as a subspace of K , so that ψ dilates φ if and only if there is a larger Hilbert space K containing H such that for all $a \in \mathcal{A}$ we have that $\varphi(a) = P_H\psi(a)|_H$ where P_H is the projection onto H .

In the case where \mathcal{A} is unital, we say that a unital representation $\rho : \mathcal{A} \rightarrow B(K)$ is *maximal* if whenever π is a unital representation dilating ρ , then in fact $\pi = \rho \oplus \psi$ for some unital representation ψ .

Muhly and Solel [33] and Ditschel and McCullough [16, Theorem 1.1] showed that a unital representation $\rho : \mathcal{A} \rightarrow B(K)$ is maximal with respect to \mathcal{A} if and only if it has the unique extension property with respect to \mathcal{A} . Ditschel and McCullough [16, Theorem 1.2] (see also [3]) then used this to show that every unital representation ρ on \mathcal{A} can be dilated to a *maximal* unital representation π on \mathcal{A} . This provided the first dilation-theoretic proof for the existence of the C^* -envelope.

2.2. Non-commutative boundaries for non-unital algebras. We explain how to define the notions of maximality and the unique extension property for representations of non-unital operator algebras, in a way that yields essentially the same theory as in the unital case.

If $\mathcal{A} \subseteq B(H)$ is a non-unital operator algebra generating a C^* -algebra \mathcal{B} , a theorem of Meyer [31, Section 3] (see also [7, Corollary 2.1.15]) states that every representation $\varphi : \mathcal{A} \rightarrow B(K)$ extends to a unital representation φ^1 on the *unitization* $\mathcal{A}^1 = \mathcal{A} \oplus \mathbb{C}I_H$ of \mathcal{A} by specifying $\varphi^1(a + \lambda I_H) = \varphi(a) + \lambda I_K$. This theorem allows one to show that every representation ϕ has a completely contractive and completely positive extension to \mathcal{B} via Arveson's extension theorem. In fact, this

is a version of Arveson's extension theorem for non-unital operator algebras. Meyer's theorem also shows that \mathcal{A} has a *unique* (one-point) unitization, in the sense that if (ι, \mathcal{B}) is a C^* -cover for the operator algebra \mathcal{A} , and $\mathcal{B} \subseteq B(H)$ is some faithful representation of \mathcal{B} , then the operator-algebraic structure on $\mathcal{A}^1 \cong \iota(\mathcal{A}) + \mathbb{C}1_H$ is independent of the C^* -cover and the faithful representation of \mathcal{B} .

Next, we discuss how to extend the notions of maximality and the unique extension property to non-unital operator algebras.

Definition 2.1. *Let $\mathcal{A} \subseteq B(H)$ be an operator algebra generating a C^* -algebra \mathcal{B} .*

- (1) *We say that a representation $\rho : \mathcal{A} \rightarrow B(K)$ has the unique extension property (UEP for short) if every completely contractive and completely positive map $\pi : \mathcal{B} \rightarrow B(K)$ extending ρ is a $*$ -representation.*
- (2) *We say that a representation $\rho : \mathcal{A} \rightarrow B(K)$ is maximal if whenever π is a representation dilating ρ , then $\pi = \rho \oplus \psi$ for some representation ψ .*

Remark 2.2. When the maps in the definitions above are not assumed multiplicative, there are instances where the UEP is satisfied vacuously. We thank Raphael Clouatre for bringing these issues to our attention.

Indeed, Suppose \mathcal{A} is a non-unital operator algebra containing a self-adjoint positive element P and let $\rho : \mathcal{A} \rightarrow \mathcal{B}$ be a completely contractive homomorphism. The map $-\rho$ is completely contractive, but cannot be extended to a completely contractive completely positive map on $\mathcal{B} = C^*(\mathcal{A})$, as ρ must send P to $-P$. Hence, $-\rho$ vacuously has the UEP. Furthermore, when ρ is *not* maximal, the map $-\rho$ is a completely contractive map that admits a non-trivial completely contractive dilation, coming from the one for ρ . Hence, $-\rho$ is also not maximal. Thus, we see that if we drop the multiplicativity assumptions in our definitions above, the UEP and maximality would not be equivalent.

By a similar proof to [3, Proposition 2.2], and by the Arveson extension theorem for non-unital operator algebras via Meyer's theorem, we get that maximality is equivalent to the UEP.

Consequently, since maximality does not depend on the choice of C^* -cover, the unique extension property for representations does not depend on the choice of C^* -cover, even for non-unital operator algebras. We will often refer to this fact as the "*invariance of the UEP*".

For a representation ρ it is easy to see that ρ is maximal if and only if ρ^1 is maximal. Hence, as maximality is equivalent to the UEP, we see

that a representation ρ on \mathcal{A} has the UEP if and only if its unitization ρ^1 has the UEP.

Suppose \mathcal{A} is an operator subalgebra of $B(H)$, and $\rho : \mathcal{A} \rightarrow B(K)$ is a representation. We can write $\rho := \rho_{nd} \oplus 0^{(\alpha)}$, where $0 : \mathcal{A} \rightarrow \mathbb{C}$ is the zero map and α is some multiplicity, such that ρ_{nd} is the non-degenerate part in the sense that $\rho_{nd}(a) = \rho(a)|_L$ with $L := C^*(\rho(\mathcal{A}))K$.

When \mathcal{A} is unital, we get that any completely contractive completely positive extension of $0 : \mathcal{A} \rightarrow \mathbb{C}$ to $\mathcal{B} = C^*(\mathcal{A})$ must be 0. As the direct sum of representations with the UEP still has the UEP, we see that ρ has the UEP if and only if the *unital* representation ρ_{nd} has the UEP.

In the case where \mathcal{A} is separable, non-unital and contains a positive approximate identity, we let $0^1 : \mathcal{A}^1 \rightarrow \mathbb{C}$ be the unitization of the zero map, which is a unital representation. Since this map extends uniquely to a map on the operator system $\mathcal{S} = \mathcal{A}^1 + (\mathcal{A}^1)^*$, which we still denote by 0^1 , and as $\mathcal{A} \cap \mathcal{A}^*$ contains a positive approximate identity, by [6, Theorem 6.1] we see that 0^1 has the UEP when restricted to \mathcal{A}^1 . Hence, the restriction $0 = 0^1|_{\mathcal{A}}$ has the UEP.

Hence, if we assume that \mathcal{A} is separable and has a positive approximate identity, we still have that ρ has UEP if and only if ρ_{nd} has UEP. These assumptions will be satisfied by all operator algebras associated with graphs in this paper, so we will not belabor this point further.

The C^* -envelope of a non-unital operator algebra can also be computed from the C^* -envelope of its unitization. More precisely, as the pair $(C_e^*(\mathcal{A}), \iota)$ where $C_e^*(\mathcal{A})$ is the C^* -subalgebra generated by $\iota(\mathcal{A})$ inside the C^* -envelope $(C_e^*(\mathcal{A}^1), \iota)$ of the (unique) unitization \mathcal{A}^1 of \mathcal{A} . By the proof of [7, Proposition 4.3.5] this C^* -envelope of an operator algebra \mathcal{A} has the desired universal property, that for any C^* -cover (ι', \mathcal{B}') of \mathcal{A} , there exists a (necessarily unique and surjective) $*$ -homomorphism $\pi : \mathcal{B}' \rightarrow C_e^*(\mathcal{A})$, such that $\pi \circ \iota' = \iota$.

As to representations with the UEP, when \mathcal{A} is an operator algebra generating a C^* -algebra \mathcal{B} , using these unitization tricks, the theorem of Ditschel and McCullough (see [3]) in the unital case shows that $C_e^*(\mathcal{A})$ is again the image of a $*$ -representation $\rho : \mathcal{B} \rightarrow B(K)$ such that $\rho|_{\mathcal{A}}$ is completely isometric and has the unique extension property.

Let \mathcal{A} be an operator algebra generating a C^* -algebra \mathcal{B} . We say that \mathcal{A} has the unique extension property in \mathcal{B} if for any faithful $*$ -representation $\pi : \mathcal{B} \rightarrow B(H)$ we have that $\pi|_{\mathcal{A}}$ has the unique extension property. By taking a direct sum of π with a given $*$ -representation of \mathcal{B} , it is easy to show that the faithfulness assumption can be dropped, and in particular, we must have that $\mathcal{B} \cong C_e^*(\mathcal{A})$.

We will need the following result on the existence of a largest subrepresentation with the UEP. Let $\phi : \mathcal{B} \rightarrow B(H)$ be a completely contractive completely positive map on a C^* -algebra \mathcal{B} , and let $K \subseteq H$ be a reducing subspace for $\phi(A)$. Let $\phi_K : \mathcal{B} \rightarrow B(K)$ denote the restriction $\phi_K(b) = \phi(b)|_K$.

Proposition 2.3. *Let \mathcal{A} be an operator algebra generating a C^* -algebra \mathcal{B} and let $\pi : \mathcal{B} \rightarrow B(H)$ be a $*$ -representation. Then there is a unique (perhaps trivial) largest reducing subspace K for π such that $\pi_K|_{\mathcal{A}}$ has the unique extension property.*

Proof. If there is no such non-trivial reducing subspace, we take $K = \{0\}$. Otherwise, let \mathcal{C} be the (non-empty) collection of non-trivial reducing subspaces L for π such that $\pi_L : \mathcal{B} \rightarrow B(L)$ has the UEP when restricted to \mathcal{A} . Set $K := \bigvee_{L \in \mathcal{C}} L$. Since every $L \in \mathcal{C}$ is reducing for π , we must have that K is reducing for π as well. It remains to show that $\pi_K|_{\mathcal{A}}$ has the UEP. To this end, let $\phi : \mathcal{B} \rightarrow B(K)$ be a completely contractive completely positive extension of $\pi_K|_{\mathcal{A}}$. Then for every $L \in \mathcal{C}$ the map $P_L\phi(\cdot)|_L$ is a completely contractive completely positive map from \mathcal{B} to $B(L)$ and $P_L\phi(a)|_L = P_L\pi_K(a)|_L = \pi_L(a)$ for every $a \in \mathcal{A}$. As π_L has the UEP we have that $P_L\phi(b)|_L = \pi_L(b)$ for every $b \in \mathcal{B}$. In addition, by Schwarz inequality, for every $L \in \mathcal{C}$ and $b \in \mathcal{B}$,

$$\begin{aligned} 0 &\leq P_L\phi(b)^*(I_K - P_L)\phi(b)P_L \\ &= P_L\phi(b)^*\phi(b)P_L - P_L\phi(b)^*P_L\phi(b)P_L \\ &\leq P_L\phi(b^*b)P_L - P_L\phi(b)^*P_L\phi(b)P_L \\ &= \pi_L(b^*b) - \pi_L(b)^*\pi_L(b) = 0 \end{aligned}$$

so that $P_L\phi(b)P_L = \phi(b)P_L$ for every $b \in \mathcal{B}$. Thus, for every $n \in \mathbb{N}$, $L_1, \dots, L_n \in \mathcal{C}$, and $\xi_1 \in L_1, \dots, \xi_n \in L_n$ we have

$$\begin{aligned} \pi_K(b) \left(\sum_{i=1}^n \xi_i \right) &= \sum_{i=1}^n \pi_K(b)\xi_i = \sum_{i=1}^n \pi_{L_i}(b)\xi_i \\ &= \sum_{i=1}^n P_{L_i}\phi(b)\xi_i = \sum_{i=1}^n \phi(b)\xi_i = \phi(b) \left(\sum_{i=1}^n \xi_i \right). \end{aligned}$$

As sums $\sum_{i=1}^n \xi_i$ are dense in K , we have that $\pi_K(b) = \phi(b)$ for every $b \in \mathcal{B}$. Hence, $\pi_K|_{\mathcal{A}}$ has the unique extension property, and K is the unique largest subspace with this property. \square

3. BOUNDARIES ARISING FROM DIRECTED GRAPHS

Let $G = (V, E, s, r)$ be a countable directed graph. We will abuse terminology and call associated $*$ -representations of either $\mathcal{T}(G)$ or $\mathcal{O}(G)$ “Cuntz-Krieger” or “full Cuntz-Krieger” if their associated TCK families are such. A (universal) TCK or CK family generating $\mathcal{T}(G)$ or $\mathcal{O}(G)$ (respectively) will usually be denoted by lowercase letters (p, s) .

There is a canonical $*$ -representation of the Toeplitz-Cuntz-Krieger graph C^* -algebra which we now describe. First, recall that a path in G is a sequence of edges $\lambda = \mu_n \cdots \mu_1$ such that $r(\mu_i) = s(\mu_{i+1})$, where we extend the range and source maps to apply for paths by specifying $r(\lambda) := r(\mu_n)$ and $s(\lambda) := s(\mu_1)$, and set $|\lambda| := n$ for the length of the path; vertices are considered as paths of length 0. We use E^\bullet to denote the collection of all paths in G of finite length.

Let $H_G := \ell^2(E^\bullet)$ be the Hilbert space with canonical standard orthonormal basis $\{\xi_\lambda\}_{\lambda \in E^\bullet}$, we define a Toeplitz-Cuntz-Krieger family (P, S) on H_G by specifying each operator on an orthonormal basis, that is, for each $v \in V$, $\mu \in E$ and $\lambda \in E^\bullet$ we define

$$P_v(\xi_\lambda) = \begin{cases} \xi_\lambda & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v \end{cases} \quad \text{and} \quad S_e(\xi_\lambda) = \begin{cases} \xi_{e\lambda} & \text{if } r(\lambda) = s(e) \\ 0 & \text{if } r(\lambda) \neq s(e) \end{cases}.$$

For every $v \in V$, consider the subspace $H_{G,v} := \ell^2(s^{-1}(v))$ with its orthonormal basis $\{\xi_\lambda\}_{s(\lambda)=v}$. Clearly, $H_{G,v}$ is reducing for (P, S) , so by the universal property of $\mathcal{T}(G)$ there exists a $*$ -representation $\pi_v : \mathcal{T}(G) \rightarrow B(H_{G,v})$ satisfying $\pi_v(p_w) = P_w|_{H_{G,v}}$ for every $w \in V$ and $\pi_v(s_e) = S_e|_{H_{G,v}}$ for every $e \in E$. The next proposition is easily verified, and we omit its proof.

Proposition 3.1. *Let $\pi_v : \mathcal{T}(G) \rightarrow B(H_{G,v})$ be the $*$ -representation described above. Then the following hold:*

- (a) π_v is irreducible,
- (b) for every $w \neq v$ we have $\text{SOT-}\sum_{r(e)=w} \pi_v(s_e s_e^*) = \pi_v(p_w)$, and
- (c) $\pi_v(p_v) - \text{SOT-}\sum_{r(e)=v} \pi_v(s_e s_e^*)$ is a rank 1 projection.

Toeplitz-Cuntz-Krieger families have the following useful version of the Wold decomposition. A slightly different Wold decomposition was given in [20, Section 2] by Jury and Kribs under the assumption that the graphs have no sinks. Here we give a self-contained, and slightly more general version, that is tailored to our context. Let V_r the set of vertices $v \in V$ such that $r^{-1}(v) \neq \emptyset$.

Theorem 3.2 (Wold decomposition). *Let (Q, T) be a Toeplitz-Cuntz-Krieger family on a Hilbert space H . For every $v \in V_r$, denote by α_v*

the dimension of the space $W_v := (Q_v - \sum_{r(e)=v} T_e T_e^*)H$. Then (Q, T) is unitarily equivalent to

$$\oplus_{v \in V_r} ((P, S)|_{H_{G,v}})^{(\alpha_v)} \oplus (R, L)$$

where (R, L) is a full CK G -family. In addition, this representation is unique in the sense that if (Q, T) is unitarily equivalent to

$$\oplus_{v \in V_r} ((P, S)|_{H_{G,v}})^{(\alpha'_v)} \oplus (R', L')$$

where (R', L') is a full CK G -family, then $\alpha'_v = \alpha_v$ for every $v \in V_r$, and (R, L) is unitarily equivalent to (R', L') .

Proof. Uniqueness follows by Proposition 3.1. Indeed, as $(P, S)|_{H_{G,v}}$ cannot be unitarily equivalent to $(P, S)|_{H_{G,w}}$ for $w \neq v$ nor to restrictions to reducing subspaces for either full CK families (R', L') or (R, L) . Thus, we must have that $((P, S)|_{H_{G,v}})^{(\alpha_v)}$ is unitarily equivalent to $((P, S)|_{H_{G,v}})^{(\alpha'_v)}$ so that $\alpha_v = \alpha'_v$. Once this is established, restricting to the orthocomplement of the (reducing) subspaces associated with $\oplus_{v \in V_r} ((P, S)|_{H_{G,v}})^{(\alpha_v)}$ and $\oplus_{v \in V_r} ((P, S)|_{H_{G,v}})^{(\alpha'_v)}$, we obtain a unitary equivalence between (R, L) and (R', L') .

As for existence, fix $v \in V_r$, and denote $W_v = (Q_v - \sum_{r(e)=v} T_e T_e^*)H$. Choose an orthonormal basis $\{\zeta_v^{(i)}\}$ for W_v , of cardinality α_v , and for every i set

$$H_{v,i} := \text{span}\{T_\lambda \zeta_v^{(i)} : \lambda \in s^{-1}(v)\}.$$

We will show these subspace are reducing. Indeed, $H_{v,i}$ is clearly invariant for the family (Q, T) . As for co-invariance, note that $T_\mu^*(T_\lambda \zeta_v^{(i)})$ is either 0, a vector of the form $T_{\lambda'} \zeta_v^{(i)}$ for some path λ' , or a vector of the form $T_{\mu'}^* \zeta_v^{(i)}$ for some path μ' with $|\mu'| \geq 1$. As the two first cases immediately imply that $T_\mu^*(T_\lambda \zeta_v^{(i)}) \in H_{v,i}$, we need to deal only with the third case. To this end, write $\mu' = e_0 \mu''$ for some edge $e_0 \in E$ and a path μ'' with $s(\mu'') = r(e_0)$. Note that if $e_0 \in s^{-1}(v)$, then $T_{e_0}^*(Q_v - \sum_{r(e)=v} T_e T_e^*) = T_{e_0}^* - T_{e_0}^* = 0$, and otherwise, $T_{e_0}^*(Q_v - \sum_{r(e)=v} T_e T_e^*) = 0 - 0 = 0$. Thus, in any case,

$$T_{\mu'}^* \zeta_v^{(i)} = T_{\mu'}^*(Q_v - \sum_{r(e)=v} T_e T_e^*) \zeta_v^{(i)} = T_{\mu''}^* T_{e_0}^*(Q_v - \sum_{r(e)=v} T_e T_e^*) \zeta_v^{(i)} = 0.$$

We next show simultaneously that for fixed $v \in V_r$ and $1 \leq i \leq \alpha_v$, the set $\{T_\lambda \zeta_v^{(i)}\}_{\lambda \in s^{-1}(v)}$ is an orthonormal family, and that the spaces $H_{v,i}$ are pairwise orthogonal for all $v \in V_r$ and $1 \leq i \leq \alpha_v$. Our first step is to show that for two vertices $v, w \in V_r$, two indices $1 \leq i \leq \alpha_v$

and $1 \leq j \leq \alpha_w$, and two paths λ, μ in G , if $\langle T_\lambda \zeta_v^{(i)}, T_\mu \zeta_w^{(j)} \rangle \neq 0$ then we must have $\lambda = \mu$. Indeed,

$$\langle T_\lambda \zeta_v^{(i)}, T_\mu \zeta_w^{(j)} \rangle = \left\langle \left((Q_w - \sum_{r(e)=w} T_e T_e^*) T_\mu^* T_\lambda (Q_v - \sum_{r(e)=v} T_e T_e^*) \right) \zeta_v^{(i)}, \zeta_w^{(j)} \right\rangle.$$

For $T_\mu^* T_\lambda$ to be non-zero, it must be either of the form $T_{\lambda'}$ where $\lambda = \mu \lambda'$, or $T_{\mu'}^*$ where $\mu = \lambda \mu'$. We deal with the first case, and the second is proven similarly. So assume $\lambda = \mu \lambda'$. If $|\lambda'| = 0$, then $\lambda = \mu$. If $|\lambda'| \geq 1$, write $\lambda' = e_0 \lambda''$. Then we have

$$(Q_w - \sum_{e \in r^{-1}(w)} T_e T_e^*) T_\mu^* T_\lambda = T_{\lambda'} - T_{e_0} T_{e_0}^* T_{\lambda'} = 0$$

which yields a contradiction. Thus, $\lambda = \mu$.

As a consequence of this, we see that $v = s(\lambda) = s(\mu) = w$. As T_λ is an isometry on $P_v H$, the assumption $\langle T_\lambda \zeta_v^{(i)}, T_\lambda \zeta_v^{(j)} \rangle \neq 0$ yields $i = j$ as well. We therefore must have that the sets $\{T_\lambda \zeta_v^{(i)} : \lambda \in s^{-1}(v)\}$ are orthonormal bases for the pairwise orthogonal reducing subspaces $H_{v,i}$.

We next define unitaries $U_{v,i} : H_{v,i} \rightarrow H_{G,v}$ by mapping an orthonormal basis to an orthonormal basis $U_{v,i} : T_\lambda \zeta_v^{(i)} \mapsto \xi_\lambda$. Clearly $U_{v,i}$ intertwines $(Q, T)|_{H_{v,i}}$ and $(P, S)|_{H_{G,v}}$. Denote by $K = (\oplus_{v \in V_r} H_{v,i})^\perp$, we then have that (Q, T) is unitarily equivalent to

$$\oplus_{v \in V_r} ((P, S)|_{H_{G,v}})^{(\alpha_v)} \oplus (Q, T)|_K.$$

As a result $(R, L) := (Q, T)|_K$ is a Topelitz-Cuntz-Krieger family such that for any $v \in V_r$ we have $R_v = \text{SOT} - \sum_{r(e)=v} L_e L_e^*$. Hence, it is a full CK family. \square

By rephrasing the previous proposition in terms of $*$ -representations, we obtain the following corollary.

Corollary 3.3. *Let G be a directed graph, and let $\pi : \mathcal{T}(G) \rightarrow B(H)$ be a $*$ -representation. Then there are multiplicities $\{\alpha_v\}_{v \in V_r}$ such that π is unitarily equivalent to the $*$ -representation $\pi_s \oplus \pi_b$, where $\pi_s = \oplus_{v \in V_r} \pi_v^{(\alpha_v)}$ and π_b is a full CK representation. In addition, this representation is unique in the sense that if π is also unitarily equivalent to the $*$ -representation $\pi'_s \oplus \pi'_b$, where $\pi'_s = \oplus_{v \in V_r} \pi_v^{(\alpha'_v)}$ and π'_b is a full CK representation, then $\alpha'_v = \alpha_v$ for every $v \in V_r$, and π'_b is unitarily equivalent to π_b .*

We next characterize those $*$ -representations which have the unique extension property with respect to $\mathcal{T}_+(G)$.

Definition 3.4. Let $G = (V, E, s, r)$ be a directed graph, and let $\pi : \mathcal{T}(G) \rightarrow B(H)$ be a $*$ -representation. We say that $v \in V_r$ is singular with respect to π (or simply that v is π -singular) if

$$\text{SOT-} \sum_{r(e)=v} \pi(S_e S_e^*) \not\leq \pi(P_v).$$

Note that π is a full CK representation of $\mathcal{T}(G)$, if and only if π has no singular vertices.

Theorem 3.5. Suppose that $\pi : \mathcal{T}(G) \rightarrow B(H)$ is a $*$ -representation. The restriction $\pi|_{\mathcal{T}_+(G)}$ has the unique extension property if and only if π is a full CK representation.

Proof. Let (p, s) be a generating TCK family for $\mathcal{T}(G)$. Suppose G has a π -singular vertex v . If we assume towards contradiction that $\pi|_{\mathcal{T}_+(G)}$ has the unique extension property, then by [6, Proposition 4.4], so does the restriction of the infinite inflation $\pi^{(\infty)} : \mathcal{T}(G) \rightarrow B(H^{(\infty)})$ to $\mathcal{T}_+(G)$. We therefore may assume without loss of generality that π has infinite multiplicity. We will arrive at a contradiction by showing that $\pi|_{\mathcal{T}_+(G)}$ is not maximal.

As v is π -singular, and π has infinite multiplicity, the projection $Q_v := \pi(p_v) - \text{SOT-} \sum_{r(e)=v} \pi(s_e s_e^*)$ is infinite dimensional. Thus, we may decompose $Q_v H = \oplus_{r(e)=v} H_e$ into infinite dimensional spaces H_e for each $e \in r^{-1}(v)$. We can then define for every $e \in r^{-1}(v)$ some isometry $W_e : P_{s(e)} H_G \rightarrow H_e$ where H_G is the Hilbert space $\ell^2(E^\bullet)$ and (P, S) is the associated TCK G -family. We moreover define a $*$ -representation $\rho : \mathcal{T}(G) \rightarrow B(H \oplus H_G)$ by specifying a Toeplitz-Cuntz-Krieger G -family

$$\rho(p_v) = \begin{bmatrix} \pi(p_v) & 0 \\ 0 & P_v \end{bmatrix} \text{ for all } v \in V$$

and

$$\rho(s_e) = \begin{cases} \begin{bmatrix} \pi(s_e) & W_e \\ 0 & 0 \end{bmatrix} & \text{if } r(e) = v, \quad \text{and} \\ \begin{bmatrix} \pi(s_e) & 0 \\ 0 & S_e \end{bmatrix} & \text{otherwise.} \end{cases}$$

We show this defines a Toeplitz-Cuntz-Krieger family. Clearly, we need to verify only those relations which involve edges in $r^{-1}(v)$. For every $e \in r^{-1}(v)$

$$\begin{aligned}\rho(s_e)^*\rho(s_e) &= \begin{bmatrix} \pi(s_e)^* & 0 \\ W_e^* & 0 \end{bmatrix} \cdot \begin{bmatrix} \pi(s_e) & W_e \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \pi(p_{s(e)}) & \pi(s_e)^*W_e \\ W_e^*\pi(s_e) & P_{s(e)} \end{bmatrix}.\end{aligned}$$

As the range of W_e is orthogonal to that of $\pi(s_e)$, we see that $\pi(s_e)^*W_e = W_e^*\pi(s_e) = 0$, so

$$\rho(s_e)^*\rho(s_e) = \rho(p_{s(e)}),$$

and condition (I) is verified. Next, for every finite subset $F \subseteq r^{-1}(v)$

$$\begin{aligned}\sum_{e \in F} \rho(s_e)\rho(s_e)^* &= \sum_{e \in F} \begin{bmatrix} \pi(s_e) & W_e \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \pi(s_e)^* & 0 \\ W_e^* & 0 \end{bmatrix} \\ &= \sum_{e \in F} \begin{bmatrix} \pi(s_e s_e^*) + W_e W_e^* & 0 \\ 0 & 0 \end{bmatrix} \\ &\leq \begin{bmatrix} \pi(p_v) & 0 \\ 0 & P_v \end{bmatrix} = \rho(p_v).\end{aligned}$$

where the inequality is true since $\{\pi(s_e s_e^*)\} \cup \{W_e W_e^*\}$ is a collection of pairwise orthogonal projections dominated by $\pi(p_v)$. We therefore have shown condition (TCK), and we conclude that $\rho|_{\mathcal{T}_+(G)}$ is a well-defined representation which dilates $\pi|_{\mathcal{T}_+(G)}$ non-trivially. Hence $\pi|_{\mathcal{T}_+(G)}$ is not maximal.

For the converse, suppose that G has no π -singular vertices. Let $\tilde{\rho} : \mathcal{T}_+(G) \rightarrow B(K)$ be a maximal dilation of $\pi|_{\mathcal{T}_+(G)}$, and let $\rho : \mathcal{T}(G) \rightarrow B(K)$ be its extension to a $*$ -representation. Denote

$$\rho(p_v) = \begin{bmatrix} \pi(p_v) & X_v \\ Y_v & Z_v \end{bmatrix} \quad \text{and} \quad \rho(s_e) = \begin{bmatrix} \pi(s_e) & X_e \\ Y_e & Z_e \end{bmatrix}$$

for all $v \in V$ and $e \in E$. We have that $X_v = 0$ and $Y_v = 0$ for all $v \in V$. Indeed, let $v \in V$, and $P : K \rightarrow H$ the orthogonal projection onto H , then

$$\begin{aligned}P\rho(p_v)^*(1-P)\rho(p_v)P &= P\rho(p_v)P - P\rho(p_v)P\rho(p_v)P \\ &= \pi(p_v) - \pi(p_v)\pi(p_v) = 0.\end{aligned}$$

and the C^* -identity implies $Y_v = (1-P)\rho(p_v)P = 0$. As $\rho(p_v)$ is self-adjoint, we have $X_v = 0$ as well.

Next, for all $e \in E$ we have $p_{s(e)} = s_e^* s_e$, so

$$\begin{bmatrix} \pi(p_{s(e)}) & 0 \\ 0 & * \end{bmatrix} = \rho(s_e^* s_e) = \rho(s_e)^* \rho(s_e) = \begin{bmatrix} \pi(s_e)^* \pi(s_e) + Y_e^* Y_e & * \\ * & * \end{bmatrix}$$

which implies $Y_e = 0$ for all $e \in E$.

Finally, let $e \in E$ and let $v = r(e)$. For every finite subset F of $r^{-1}(v)$, we have $p_v \geq \sum_{f \in F} s_f s_f^*$, so

$$\begin{aligned} \begin{bmatrix} \pi(p_v) & 0 \\ 0 & * \end{bmatrix} &= \rho(p_v) \geq \sum_{f \in F} \rho(s_f) \rho(s_f)^* \\ &= \sum_{f \in F} \begin{bmatrix} \pi(s_f) \pi(s_f)^* + X_f X_f^* & * \\ * & * \end{bmatrix}. \end{aligned}$$

In particular, by compressing this inequality to H we obtain

$$\sum_{f \in F} \pi(s_f) \pi(s_f)^* + X_f X_f^* \leq \pi(p_v)$$

for every finite subset F of $r^{-1}(v)$. Since v is not π -singular, we must have that

$$\sup_F \sum_{f \in F} \pi(s_f) \pi(s_f)^* = \text{SOT-} \sum_{r(f)=v} \pi(s_f) \pi(s_f)^* = \pi(p_v).$$

We therefore obtain that $X_f = 0$ for all $f \in r^{-1}(v)$, and in particular $X_e = 0$. Since $\mathcal{T}(G)$ is generated as a C^* -algebra by $\mathcal{T}_+(G)$, we must have that ρ has π as a direct summand, and hence $\rho|_{\mathcal{T}_+(G)}$ is a trivial dilation of $\pi|_{\mathcal{T}_+(G)}$. \square

The previous theorem gives rise to two interesting corollaries. The first is a parametrization of those irreducible $*$ -representations of $\mathcal{T}(G)$ which are not boundary representations with respect to $\mathcal{T}_+(G)$, and the second is the dilation of TCK families to full CK families.

Corollary 3.6. *For every vertex $v \in V_r$, the $*$ -representation $\pi_v : \mathcal{T}(G) \rightarrow B(H_{G,v})$ is the unique irreducible $*$ -representation (up to unitary equivalence) for which v is π -singular, so that the irreducible $*$ -representations of $\mathcal{T}(G)$ which are not boundary for $\mathcal{T}_+(G)$ are parametrized by V_r .*

Proof. If π is an irreducible $*$ -representation that lacks the unique extension property on $\mathcal{T}_+(G)$, then by Theorem 3.5 there exists $v \in V_r$ which is π -singular. By the Wold decomposition (Corollary 3.3), up to a unitary equivalence, π must have π_v as a subrepresentation, and by irreducibility, π is unitarily equivalent to π_v . \square

Corollary 3.7. *Let $G = (V, E, s, r)$ be a countable directed graph, and (P, S) a TCK family on H . Then there exists a full CK family (Q, T) on a Hilbert space K containing H , such that $f(P, S) = P_H f(Q, T)|_H$ for any polynomial $f \in \mathbb{C}\langle V, E \rangle$ in non-commuting variables.*

Proof. Let $\pi_{P,S} : \mathcal{T}(G) \rightarrow B(H)$ be the $*$ -representation of $\mathcal{T}(G)$ associated to (P, S) . By [16, Theorem 1.2] we can dilate $\pi_{P,S}|_{\mathcal{T}_+(G)}$ to a maximal representation $\tau : \mathcal{T}_+(G) \rightarrow B(K)$, and without loss of generality, H is a subspace of K . Hence, τ is the restriction to $\mathcal{T}_+(G)$ of a $*$ -representation $\rho : \mathcal{T}(G) \rightarrow B(K)$ such that $\rho|_{\mathcal{T}_+(G)}$ has the unique extension property. Let (Q, T) be the TCK family associated to τ . By Theorem 3.5 (Q, T) is a full CK family, and it dilates (P, S) in the sense that for every polynomial $f \in \mathbb{C}\langle V, E \rangle$ we have that $f(P, S) = P_H f(Q, T)|_H$. \square

Our next goal is to construct, for every directed graph G , faithful full CK representations of $\mathcal{O}(G)$. We do this by constructing certain universal CK families arising from backward-infinite paths.

Let $E^\infty = \{ \lambda \mid \lambda = e_1 e_2 e_3 \cdots, s(e_i) = r(e_{i+1}), e_i \in E \}$ be the collection of all backward infinite paths in G , and extend the range map to E^∞ by setting $r(\lambda) = r(e_1)$ for $\lambda = e_1 e_2 e_3 \cdots \in E^\infty$. Let $E^{<\infty}$ be the collection of all finite paths, including paths of length 0, emanating from a source, and set $E^{\leq\infty} = E^\infty \cup E^{<\infty}$ as a disjoint union.

For each vertex $v \in V$ fix an element $\mu_v \in E^{\leq\infty}$ with $r(\mu_v) = v$. For $i \in \mathbb{N}$ define $\mu_{v,i} = e_{v,1} \dots e_{v,i}$ as the i -th truncation of μ_v , where if $|\mu_v| \leq i$, $\mu_{v,i} = \mu_v$. Denote $H_{v,i}$ the Hilbert space with the orthonormal basis $\{ \xi_{\lambda \mu_{v,i}^{-1}} \mid \lambda \in E^\bullet \}$ where $\lambda \mu_{v,i}^{-1}$ corresponds to the equivalence class of reduced products determined by λ and $\mu_{v,i}$. We set $\Gamma := \{ \lambda \mu_{v,i}^{-1} \mid \lambda \in E^\bullet, i \in \mathbb{N}, v \in V \}$, and let $H_b := \ell^2(\Gamma)$ denote the Hilbert space with orthonormal basis $\{ \xi_{\lambda \mu^{-1}} \}_{\lambda \mu^{-1} \in \Gamma}$, which is unitarily equivalent to $\bigoplus_{v \in V} \left[\bigvee_{i \in \mathbb{N}} H_{v,i} \right]$, where $H_{v,i}$ is identified as a subspace of $H_{v,i+1}$ since $\lambda \mu_{v,i}^{-1}$ is identified with $\lambda e_{v,i+1} \mu_{v,i+1}^{-1}$ whenever $|\mu_v| > i$. We define a TCK family (Q, T) on H_b by specifying it on the orthonormal basis $\{ \xi_{\lambda \mu^{-1}} \}_{\lambda \mu^{-1} \in \Gamma}$ by

$$Q_v(\xi_{\lambda \mu^{-1}}) = \begin{cases} \xi_{\lambda \mu^{-1}} & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v \end{cases}, \quad T_e(\xi_{\lambda \mu^{-1}}) = \begin{cases} \xi_{e \lambda \mu^{-1}} & \text{if } r(\lambda) = s(e) \\ 0 & \text{if } r(\lambda) \neq s(e) \end{cases}.$$

It is easy to verify that (Q, T) is a full CK family. Indeed, $\xi_{\lambda \mu_{v,i}^{-1}}$, with $|\lambda| > 1$ is in the range of T_e for $e \in E$ such that $\lambda = e \lambda'$, and each $\xi_{\mu_{v,i}^{-1}}$ where $s(\mu) = s(\mu_{v,i})$ is not a source is in the range of $T_{e_{v,i+1}}$ as $\mu_{v,i}^{-1}$ is identified with $e_{v,i+1} \mu_{v,i+1}^{-1}$.

By construction, each Q_v is non-zero for all $v \in V$, and we let ρ_∞ be the $*$ -representation of $\mathcal{T}(G)$ associated to (Q, T) above. Moreover, by construction of ρ_∞ , for each $z \in \mathbb{T}$ we get a well-defined unitary $U_z : H_b \rightarrow H_b$ by specifying $U_z(\xi_{\lambda \mu^{-1}}) = z^{|\mu| - |\lambda|} \xi_{\lambda \mu^{-1}}$. It is then easy

to see that $U_z Q_v U_z^* = Q_v$ and $U_z T_e U_z^* = z T_e$ so we may define a gauge action $\alpha : \mathbb{T} \rightarrow C^*(Q, T)$ via $\alpha_z(A) = U_z A U_z^*$, so that $\alpha_z(Q_v) = Q_v$ and $\alpha_z(T_e) = z T_e$. Hence, by the gauge invariant uniqueness theorem ρ_∞ is injective. The advantage of this construction is that it produces a space which has a natural action of \mathbb{T} on it.

Remark 3.8. *One may form a full CK family on $\ell^2(E^{\leq \infty})$ in a similar way, but this representation will fail to be injective when the graph has a vertex-simple cycle with no entry.*

Let $\mathcal{J}(G)$ denote the kernel of the quotient $q : \mathcal{T}(G) \rightarrow \mathcal{O}(G)$. Evidently, $\mathcal{J}(G)$ is the ideal of $\mathcal{T}(G)$ generated by terms of the form $p_v - \sum_{r(e)=v} s_e s_e^*$ for vertices v with $0 < |r^{-1}(v)| < \infty$. In [21, Theorem 3.3], Kakariadis showed that $\mathcal{T}_+(G)$ has the unique extension property in $\mathcal{O}(G)$ when G is row-finite. We provide the proof for this statement along with its converse, and the computation of the C^* -envelope.

Theorem 3.9. *Let $G = (V, E, s, r)$ be a directed graph, and let $q : \mathcal{T}(G) \rightarrow \mathcal{O}(G)$ be the natural quotient map. Then q is completely isometric on $\mathcal{T}_+(G)$, and $C_e^*(\mathcal{T}_+(G)) \cong \mathcal{O}(G)$.*

Moreover, $\mathcal{T}_+(G)$ has the unique extension property in $\mathcal{O}(G)$ if and only if G is row-finite.

Proof. Let (p, s) be a TCK family such that its associated $*$ -representation $\pi_{p,s}$ is faithful. By [16, Theorem 1.1] we know that $\pi_{p,s}|_{\mathcal{T}_+(G)}$ has a maximal dilation, so let $\tau : \mathcal{T}(G) \rightarrow B(K)$ be a $*$ -representation such that $\tau|_{\mathcal{T}_+(G)}$ is the dilation of $\pi_{p,s}|_{\mathcal{T}_+(G)}$, so that it is completely isometric and with the unique extension property. By Theorem 3.5, τ is a full CK representation, and hence annihilates the Cuntz-Krieger ideal $\mathcal{J}(G)$. Hence, τ must factor through the quotient map $q : \mathcal{T}(G) \rightarrow \mathcal{O}(G)$ by $\mathcal{J}(G)$, and we have that q is completely isometric on $\mathcal{T}_+(G)$.

Next, we show that if G is row-finite then $\mathcal{T}_+(G)$ has the unique extension property in $\mathcal{O}(G)$ via q . By Theorem 3.5, we see that every $*$ -representation of $\mathcal{T}(G)$ that annihilates $\mathcal{J}(G)$ has the unique extension property when restricted to $\mathcal{T}_+(G)$ inside $\mathcal{T}(G)$. Since q is completely isometric on $\mathcal{T}_+(G)$, by invariance of the unique extension property, we see that every $*$ -representation of $\mathcal{O}(G)$ has unique extension property when restricted to $\mathcal{T}_+(G)$ inside $\mathcal{O}(G)$. By [16, Theorem 1.1] we know that the C^* -envelope of $\mathcal{T}_+(G)$ is the image under the direct sum of all $*$ -representations of $\mathcal{O}(G)$ with the unique extension property, so that $C_e^*(\mathcal{T}_+(G)) \cong \mathcal{O}(G)$ when G is row-finite, as all $*$ -representations of $\mathcal{O}(G)$ have the unique extension property when restricted to $\mathcal{T}_+(G)$ inside $\mathcal{O}(G)$.

Otherwise, if G is not row-finite, we have that $\rho_\infty \circ q$ is a full CK representation with kernel $\mathcal{J}(G)$, and hence again by invariance of the unique extension property and Theorem 3.5 we have that ρ_∞ has the unique extension property on $\mathcal{T}_+(G)$. Hence, since ρ_∞ is faithful, we still have that $C_e^*(\mathcal{T}_+(G)) \cong \mathcal{O}(G)$.

For the converse of the second part of the statement, suppose that G is not row-finite, and let $v \in V$ be an infinite receiver. Then π_v annihilates $\mathcal{J}(G)$, so we may consider the induced $*$ -representation $\dot{\pi}_v : \mathcal{O}(G) \rightarrow B(H_{G,v})$. By Corollary 3.6 π_v does not have the unique extension property when restricted to $\mathcal{T}_+(G)$, so that by invariance of the unique extension property, $\dot{\pi}_v$ does not have the unique extension property when restricted to $\mathcal{T}_+(G)$. Thus, $\mathcal{T}_+(G)$ does not have the unique extension property in $\mathcal{O}(G)$. \square

Remark 3.10. In Theorem 3.5, and Corollaries 3.6 and 3.7 we avoided the use of a uniqueness theorem. This is also true for the computation of the C^* -envelope in Theorem 3.9 when G is row-finite. A uniqueness theorem was needed only for the computation of the C^* -envelope when the graph is non-row-finite.

4. FREE PRODUCTS AND THE UNIQUE EXTENSION PROPERTY

Consider the category of unital \mathbb{C} -algebras (with unital homomorphisms as morphisms). Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of unital \mathbb{C} -algebras and let \mathcal{D} be a common unital subalgebra with $1_{\mathcal{A}_i} \in \mathcal{D}$, let $\iota_i : \mathcal{D} \rightarrow \mathcal{A}_i$ denote the natural embeddings. Pushouts in this category are known to exist, and are called *free product of $\{\mathcal{A}_i\}_{i \in I}$ amalgamated over the common subalgebra \mathcal{D}* , denoted by $*_{\mathcal{D}} \mathcal{A}_i$. We recall the details briefly.

We let $*\mathcal{A}_i$ (with no \mathcal{D}) denote the vector space spanned by formal expressions $a_1 * \cdots * a_n$ where $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$ such that $i_1 \neq i_2 \neq \cdots \neq i_n$ and $n \geq 1$, where this expression behaves multilinearly, and we define multiplication of two such expressions, where $b_1 \in \mathcal{A}_{j_1}, \dots, b_m \in \mathcal{A}_{j_m}$ with $j_1 \neq \cdots \neq j_m$ via

$$(a_1 * \cdots * a_n) \cdot (b_1 * \cdots * b_m) = \begin{cases} a_1 * \cdots * a_n * b_1 * \cdots * b_m, & \text{if } i_n \neq j_1, \\ a_1 * \cdots * (a_n \cdot b_1) * \cdots * b_m, & \text{if } i_n = j_1. \end{cases}$$

With this multiplication, $*\mathcal{A}_i$ becomes a \mathbb{C} -algebra generated by $\{\mathcal{A}_i\}_{i \in I}$. Next, we identify the different copies of \mathcal{D} by taking a quotient by the ideal $\langle \iota_i(d) - \iota_j(d) \rangle_{i,j \in I, d \in \mathcal{D}}$. This quotient, which is denoted by $*_{\mathcal{D}} \mathcal{A}_i$, has the following universal property. If \mathcal{B} is another unital \mathbb{C} -algebra with unital \mathbb{C} -homomorphisms $\psi_i : \mathcal{A}_i \rightarrow \mathcal{B}$ which agree on \mathcal{D} , then

there is a unital \mathbb{C} -homomorphism ${}_{\mathcal{D}}^*\psi_i : {}_{\mathcal{D}}^*\mathcal{A}_i \rightarrow B$ extending each ψ_i on \mathcal{A}_i .

Next, we construct free products in the category of unital operator algebras with unital completely contractive homomorphisms. Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of unital operator algebras with \mathcal{D} a common unital operator subalgebra with $1_{\mathcal{A}_i} = 1_{\mathcal{D}}$ for all $i \in I$, and let ${}_{\mathcal{D}}^*\mathcal{A}_i$ denote the free product of $\{\mathcal{A}_i\}_{i \in I}$ amalgamated over \mathcal{D} in the larger category of \mathbb{C} -algebras. We define matrix semi-norms

$$\|a\|_n := \sup \left\| {}_{\mathcal{D}}^*\psi_i^{(n)}(a) \right\|_{B(H)}, \quad \forall n \in \mathbb{N}, a \in M_n \left({}_{\mathcal{D}}^*\mathcal{A}_i \right)$$

where the supremum is taken over all families $\{\psi_i : \mathcal{A}_i \rightarrow B(H)\}_{i \in I}$ of unital completely contractive homomorphisms that agree on \mathcal{D} and over a Hilbert space H of large enough cardinality. It follows that $\mathcal{J} = \{a \in {}_{\mathcal{D}}^*\mathcal{A}_i : \|a\| = 0\}$ is a two-sided ideal, and we denote the norms on the quotient by $\|\cdot\|_n$ as well. The norms $\|\cdot\|_n$ then define an operator-algebraic structure on the completion $\hat{{}_{\mathcal{D}}^*\mathcal{A}_i}$ of ${}_{\mathcal{D}}^*\mathcal{A}_i/\mathcal{J}$, by the Blecher-Ruan-Sinclair theorem [9]. Furthermore, by construction there are unital completely contractive homomorphisms $\iota_j : \mathcal{A}_j \rightarrow \hat{{}_{\mathcal{D}}^*\mathcal{A}_i}$. We will show in Corollary 4.2 that each ι_j is in fact completely isometric, so that each \mathcal{A}_j can be thought of as an operator subalgebra of $\hat{{}_{\mathcal{D}}^*\mathcal{A}_i}$ via ι_j .

The operator algebra $\hat{{}_{\mathcal{D}}^*\mathcal{A}_i}$ is called the free product of $\{\mathcal{A}_i\}_{i \in I}$ amalgamated over the common operator subalgebra \mathcal{D} , and has the following universal property by construction: for any unital operator algebra \mathcal{B} and $\psi_i : \mathcal{A}_i \rightarrow \mathcal{B}$ unital completely contractive homomorphisms which agree on \mathcal{D} , there exists a completely contractive homomorphism $\psi := {}_{\mathcal{D}}^*\psi_i$ from $\hat{{}_{\mathcal{D}}^*\mathcal{A}_i}$ into \mathcal{B} such that $\psi_i = \psi \circ \iota_i$.

We now provide a joint completely contractive extension result for free products of operator algebras amalgamated over a common C^* -subalgebra. Our proof is an adaptation of a proof given by Ozawa in [34, Theorem 15] in the case of amalgamation over the complex numbers. Recall that whenever \mathcal{D} is a subalgebra of an operator algebra \mathcal{A} , a completely contractive map $\phi : \mathcal{A} \rightarrow B(H)$ is said to be a \mathcal{D} -bimodule map if $\phi(a_1 d a_2) = \phi(a_1) \phi(d) \phi(a_2)$ for every $a_1, a_2 \in \mathcal{A}$ and $d \in \mathcal{D}$. By [11, Proposition 1.5.7] the restriction of a completely contractive map ϕ to a C^* -subalgebra \mathcal{D} is multiplicative if and only if ϕ is a \mathcal{D} -bimodule map.

Theorem 4.1. *Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of unital operator algebras containing a common C^* -algebra \mathcal{D} with $1_{\mathcal{A}_i} \in \mathcal{D}$, and let $\phi_i : \mathcal{A}_i \rightarrow B(H)$ be unital completely contractive \mathcal{D} -bimodule maps that agree on \mathcal{D} . Then there exists a unital completely contractive map $\phi : \hat{*}_{\mathcal{D}} \mathcal{A}_i \rightarrow B(H)$ such that $\phi \circ \iota_i = \phi_i$ for all $i \in I$.*

Proof. We construct multiplicative dilations of ϕ_i which agree on \mathcal{D} , so that the compression of their free product to H yields a unital completely contractive joint extension ϕ as in the statement of the theorem.

First, we set $H_1 := H$ and $\phi_i^{(1)} := \phi_i$. By the Arveson-Stinespring dilation theorem we may dilate each of these to a completely contractive homomorphism from \mathcal{A}_i to $B(H_1 \oplus K_1^{(i)})$. Denote by $\rho_i^{(1)}$ this dilation of ϕ_i , so that $\phi_i(a) = P_{H_1} \rho_i^{(1)}(a)|_{H_1}$. We note that since \mathcal{D} is a C^* -algebra, and each $\phi_i^{(1)}$ is multiplicative on \mathcal{D} , the space $K_1^{(i)}$ is reducing for $\rho_i^{(1)}|_{\mathcal{D}}$, so that for all $d \in \mathcal{D}$ we have

$$\rho_i^{(1)}(d) = \phi_i^{(1)}(d) \oplus P_{K_1^{(i)}} \rho_i^{(1)}(d)|_{K_1^{(i)}}.$$

Now suppose we have a sequence of subspaces

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n$$

such that for all $1 \leq m \leq n$ we have unital completely contractive \mathcal{D} -bimodule maps $\phi_i^{(m)} : \mathcal{A}_i \rightarrow B(H_m)$ that agree on \mathcal{D} , along with multiplicative unital completely contractive maps $\rho_i^{(m)} : \mathcal{A}_i \rightarrow B(H_m \oplus K_m^{(i)})$ that dilate each $\phi_i^{(m)}$, so that for all $d \in \mathcal{D}$ we have

$$\rho_i^{(m)}(d) = \phi_i^{(m)}(d) \oplus P_{K_m^{(i)}} \rho_i^{(m)}(d)|_{K_m^{(i)}}$$

and for every $j \in I$ the sequence of subspaces

$$H_1 \oplus K_1^{(j)} \subseteq H_2 \oplus K_2^{(j)} \subseteq \cdots \subseteq H_m \oplus K_m^{(j)}$$

is a sequence of reducing subspaces for $\rho_j^{(m)}$.

Denote by $H_{n+1} = H_n \oplus \bigoplus_{i \in I} K_n^{(i)}$. Fix $i \in I$, and consider the map $\tau_i : \mathcal{D} \rightarrow B(K_n^{(i)})$ given by $\tau_i(d) = P_{K_n^{(i)}} \rho_i^{(n)}(d)|_{K_n^{(i)}}$. By applying Arveson's extension theorem, followed by a restriction, for any $j \in I$ distinct from i , we may extend τ_i to a unital completely contractive map $\sigma_{ji} : \mathcal{A}_j \rightarrow B(K_n^{(i)})$. We define for all $j \in I$,

$$(4.1) \quad \phi_j^{(n+1)} := \rho_j^{(n)} \oplus \bigoplus_{j \neq i \in I} \sigma_{ji} : \mathcal{A}_j \rightarrow B(H_{n+1}),$$

so that $H_n \oplus K_n^{(j)}$ is a reducing subspace for $\phi_j^{(n+1)}$. We then have for every $d \in \mathcal{D}$ that

$$\phi_j^{(n+1)}(d) = \phi_j^{(n)}(d) \oplus \bigoplus_{i \in I} P_{K_n^{(i)}} \rho_i^{(n)}(d)|_{K_n^{(i)}}.$$

Hence, since the maps $\{\phi_i^{(n)}\}_{i \in I}$ all agree on \mathcal{D} , we have that the maps $\{\phi_i^{(n+1)}\}_{i \in I}$ all agree on \mathcal{D} .

We use Arveson's extension, Stinespring's theorem and the special form of $\phi_j^{(n+1)}$ in equation (4.1) to obtain a *multiplicative* unital completely contractive map $\rho_j^{(n+1)} : \mathcal{A}_j \rightarrow B(H_{n+1} \oplus K_{n+1}^{(j)})$ dilating $\phi_j^{(n+1)}$ such that each $H_n \oplus K_n^{(j)}$ is a reducing subspace for $\rho_j^{(n+1)}$. Hence, we then get that $\rho_j^{(n+1)}(a)|_{H_m \oplus K_m^{(j)}} = \rho_j^{(m)}(a)$ for all $1 \leq m \leq n$, and that for all $d \in \mathcal{D}$,

$$\rho_i^{(n+1)}(d) = \phi_i^{(n+1)}(d) \oplus P_{K_{n+1}^{(j)}} \rho_i^{(n+1)}(d)|_{K_{n+1}^{(j)}}.$$

Since for each $j \in I$ and $n \in \mathbb{N}$ we have $H_n \subseteq H_n \oplus K_n^{(j)} \subseteq H_{n+1}$, we may define a multiplicative unital completely contractive map $\rho_j : \mathcal{A}_j \rightarrow B(K)$ on the inductive limit of Hilbert spaces $K = \bigvee_{n \in \mathbb{N}} H_n$ by specifying $\rho_j(a)h = \rho_j^{(n)}(a)h$ for $h \in H_n \oplus K_n^{(j)}$. These maps then agree on \mathcal{D} , since for $h \in H_n$ we have that

$$\begin{aligned} \rho_i(d)h &= \rho^{(n+1)}(d)h \\ &= (\phi_i^{(n+1)}(d) \oplus P_{K_{n+1}^{(i)}} \rho_i^{(n+1)}(d)|_{K_{n+1}^{(i)}})h \\ &= \phi_i^{(n+1)}(d)h \end{aligned}$$

and as the maps $\{\phi_i^{(n+1)}\}_{i \in I}$ all agree on \mathcal{D} and the union of H_n is dense in K , we have that $\rho_j(d) = \rho_i(d)$ for all $i \neq j$ in I .

Hence, we may form the free product $\rho := \hat{*}_{\mathcal{D}} \rho_i : \hat{*}_{\mathcal{D}} \mathcal{A}_i \rightarrow B(K)$ which satisfies $\rho \circ \iota_i = \rho_i$, and the compression of ρ to H would yield a joint unital completely contractive extension ϕ as in the statement of the theorem. \square

The following allows us to identify \mathcal{A}_i as a unital operator subalgebra of $\hat{*}_{\mathcal{D}} \mathcal{A}_i$ via ι_i .

Corollary 4.2. *Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of unital operator algebras containing a common unital C^* -algebra \mathcal{D} with $1_{\mathcal{A}_i} = 1_{\mathcal{D}}$. Then for each $i \in I$ the map $\iota_i : \mathcal{A}_i \rightarrow \hat{*}_{\mathcal{D}} \mathcal{A}_i$ is completely isometric. Hence, $\hat{*}_{\mathcal{D}} \mathcal{A}_i$ is the pushout of $\{\mathcal{A}_i\}_{i \in I}$ by \mathcal{D} in the category of unital operator algebras with unital completely contractive homomorphisms.*

Proof. Fix $j \in I$ and let $\phi_j : \mathcal{A}_j \rightarrow B(H)$ be a unital completely isometric homomorphism. We may then restrict it to \mathcal{D} and use Arveson's extension theorem to extend to unital completely contractive maps $\phi_i : \mathcal{A}_i \rightarrow B(H)$ for $i \in I$ such that $i \neq j$. By Theorem 4.1 there is a joint unital completely contractive map $\phi : \hat{*}_{\mathcal{D}} \mathcal{A}_i \rightarrow B(H)$ which we may then dilate to a multiplicative map $\rho : \hat{*}_{\mathcal{D}} \mathcal{A}_i \rightarrow B(K)$. However, the compression of $\phi \circ \iota_j$ to H coincides with ϕ_j , which is a unital completely isometric map. Hence, ι_j is completely isometric. \square

In [8, Section 4], Blecher and Paulsen prove the complete injectivity of the free product of operator algebras amalgamated over the complex numbers. We next prove this where the amalgamation is over any common C^* -algebra. This generalizes [2, Proposition 2.2] due to Armstrong, Dykema, Exel and Li for free products of finitely many C^* -algebras amalgamated over a common C^* -algebra.

Proposition 4.3. *The free product of unital operator algebras amalgamated over a common C^* -subalgebra is completely injective. That is, if $\{\mathcal{A}_i\}_{i \in I}$ and $\{\mathcal{B}_i\}_{i \in I}$ are two families of unital operator algebras containing a common C^* -subalgebra \mathcal{D} such that \mathcal{A}_i is an operator subalgebra of \mathcal{B}_i for every $i \in I$ and $1_{\mathcal{A}_i} = 1_{\mathcal{B}_i} = 1_{\mathcal{D}}$, then the inclusion $\hat{*}_{\mathcal{D}} \mathcal{A}_i \subseteq \hat{*}_{\mathcal{D}} \mathcal{B}_i$ is completely isometric.*

Proof. Denote $\hat{\mathcal{A}} := \hat{*}_{\mathcal{D}} \mathcal{A}_i$ and $\hat{\mathcal{B}} := \hat{*}_{\mathcal{D}} \mathcal{B}_i$. Let $\iota_i : \mathcal{B}_i \rightarrow \hat{\mathcal{B}}$ and $\kappa_i : \mathcal{A}_i \rightarrow \hat{\mathcal{A}}$ denote the natural completely isometric inclusions. Then the unital completely isometric homomorphisms $\iota_i|_{\mathcal{A}_i} : \mathcal{A}_i \rightarrow \hat{\mathcal{B}}$ agree on \mathcal{D} , so $\phi := \hat{*}_{\mathcal{D}}(\iota_i|_{\mathcal{A}_i}) : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$ is a unital completely contractive homomorphism. Denote by $\|\cdot\|_{\hat{\mathcal{B}},n}$ and $\|\cdot\|_{\hat{\mathcal{A}},n}$ the n norms on $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ respectively.

Evidently, it would suffice to show that $\|A\|_{\hat{\mathcal{A}},n} \leq \|A\|_{\hat{\mathcal{B}},n}$ for every $A \in M_n(\hat{\mathcal{A}})$. To this end, represent $\hat{\mathcal{A}}$ completely isometrically as a unital subalgebra of $B(H)$ for some Hilbert space H . By Arveson's extension theorem the maps $\kappa_i : \mathcal{A}_i \rightarrow \hat{\mathcal{A}} \subseteq B(H)$ extend to unital completely contractive maps $\tilde{\kappa}_i : \mathcal{B}_i \rightarrow B(H)$ which agree on \mathcal{D} . By Theorem 4.1, there exists a unital completely contractive map $\psi : \hat{\mathcal{B}} \rightarrow B(H)$ such that $\psi(\iota_i(b_i)) = \tilde{\kappa}_i(b_i)$ for all $b_i \in \mathcal{B}_i$ which is completely isometric on $\hat{\mathcal{A}}$. Hence, by Stinespring's dilation theorem, we may dilate it to a unital completely contractive homomorphism $\hat{\psi} : \hat{\mathcal{B}} \rightarrow B(K)$. So for $A \in M_n(\hat{\mathcal{A}})$, we have

$$\|A\|_{\hat{\mathcal{A}},n} = \|\psi^{(n)}(A)\| \leq \|\hat{\psi}^{(n)}(A)\| \leq \|A\|_{\hat{\mathcal{B}},n}.$$

□

When $\{\mathcal{A}_i\}_{i \in I}$ are all non-unital with a common non-unital C^* -algebra \mathcal{D} , we define their free product $\hat{*}_{\mathcal{D}} \mathcal{A}_i$ to be the operator algebra generated by the images of \mathcal{A}_i inside the free product of their unitization $\hat{*}_{\mathcal{D}^1} \mathcal{A}_i^1$ amalgamated over the unitization \mathcal{D}^1 . By Meyer's theorem and the proof of [2, Lemma 2.3], we similarly get that $\hat{*}_{\mathcal{D}^1} \mathcal{A}_i^1$ coincides with the unitization $(\hat{*}_{\mathcal{D}} \mathcal{A}_i)^1$. Using this, it follows that the non-unital free product shares the analogous pushout universal property and complete injectivity described above, just as in the unital case.

Using complete injectivity, for operator algebra $\{\mathcal{A}_i\}_{i \in I}$ with a common C^* -subalgebra \mathcal{D} we can freely identify $\hat{*}_{\mathcal{D}} \mathcal{A}_i$ as a subalgebra of $\hat{*}_{\mathcal{D}} \mathcal{B}_i$, where \mathcal{B}_i is any C^* -cover for \mathcal{A}_i . We henceforth abuse notations and denote $*_{\mathcal{D}} \mathcal{A}_i$ instead of $\hat{*}_{\mathcal{D}} \mathcal{A}_i$.

This complete injectivity result for free products of not-necessarily unital operator algebras was used implicitly in [17, 18] by Duncan to show that the free product of graph tensor algebras embeds inside the free product of associated Toeplitz and Cuntz-Krieger algebras. In [13, Theorem 5.3.20] Davidson, Kakariadis and Fuller filled a gap introduced by Duncan in [18, Section 3, Theorem 1], and proved his claim [13, Theorem 5.3.21]. Our approach provides an alternative way of addressing issues of this sort.

Next, we describe a joint unital completely positive extension in the context of free products of C^* -algebras, with a special multiplicative property due to Boca.

Suppose $\{\mathcal{B}_i\}_{i \in I}$ is a family of unital C^* -algebras containing a common C^* -subalgebra \mathcal{D} with $1_{\mathcal{B}_i} \in \mathcal{D}$, and let $E_i : \mathcal{B}_i \rightarrow \mathcal{D}$ be conditional expectations. Then $\mathcal{B}_i = \mathcal{D} \oplus \ker E_i$ where the sum is in the \mathcal{D} -bimodule sense. Denote $\mathcal{B}_i^0 := \ker E_i$. Then as \mathcal{D} -bimodules we have that

$$*_{\mathcal{D}} \mathcal{B}_i = \mathcal{D} \oplus \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \dots \neq i_n} \mathcal{B}_{i_1}^0 \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} \mathcal{B}_{i_n}^0.$$

In [10, Theorem 3.1], Boca shows that if $\phi_i : \mathcal{B}_i \rightarrow B(H)$ are \mathcal{D} -bimodule unital completely positive maps that agree on \mathcal{D} , then there is a \mathcal{D} -bimodule unital completely positive map $\phi : *_{\mathcal{D}} \mathcal{B}_i \rightarrow B(H)$ with the additional multiplicative property that for $b_1 \in \mathcal{B}_{i_1}^0, \dots, b_n \in \mathcal{B}_{i_n}^0$ with $i_1 \neq i_2 \neq \dots \neq i_n$ we have

$$\phi(b_1 * \dots * b_n) = \phi_{i_1}(b_1) \dots \phi_{i_n}(b_n).$$

Proposition 4.4. *Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of either all unital, or all non-unital operator-algebras, each generating a C^* -algebra \mathcal{B}_i and containing a common C^* -algebra \mathcal{D} .*

- (1) *If $\pi_i : \mathcal{B}_i \rightarrow B(H)$ are $*$ -representations that agree on \mathcal{D} such that $\pi_i|_{\mathcal{A}_i}$ has the unique extension property when restricted to \mathcal{A}_i , then the restriction of their free product $*\pi_i$ to $*\mathcal{A}_i$ has the unique extension property.*
- (2) *If additionally we have conditional expectations $E_i : \mathcal{B}_i \rightarrow \mathcal{D}$, such that $\mathcal{A}_i = \mathcal{D} \oplus (\mathcal{A}_i \cap \ker E_i)$ as a \mathcal{D} -bimodule, then we have the converse. That is, if the restriction of a $*$ -representation $\pi : *\mathcal{B}_i \rightarrow B(H)$ to $*\mathcal{A}_i$ has the unique extension property, then for every $i \in I$ the restrictions of the $*$ -representations $\pi_i := \pi \circ \iota_i : \mathcal{B}_i \rightarrow B(H)$ to \mathcal{A}_i has the unique extension property.*

Proof. We prove the non-unital case, where the unital case is easier.

To show (1), suppose that for every i the map $\pi_i|_{\mathcal{A}_i}$ has the unique extension property. Denote $\mathcal{A} = *\mathcal{A}_i$ and $\pi := *\pi_i$. Suppose τ is a completely contractive completely positive extension of $\pi|_{\mathcal{A}}$. Then each $\tau_i := \tau \circ \iota_i$ is a completely contractive completely positive extension of $\pi_i|_{\mathcal{A}_i}$, hence $\tau_i = \pi_i$. Since each \mathcal{B}_i belongs to the multiplicative domain of τ , by [11, Proposition 1.5.7], the multiplicative domain of τ is a $*$ -subalgebra of $*\mathcal{B}_i$ generated by $\{\mathcal{B}_i\}_{i \in I}$ and hence must be equal to $*\mathcal{B}_i$, so that $\tau = \pi$.

Next, to show (2), let $\tau_i : \mathcal{B}_i \rightarrow B(H)$ be a completely contractive completely positive extension of $\pi_i|_{\mathcal{A}_i}$. By [11, Proposition 1.5.7] we have that \mathcal{D} is in the multiplicative domain of τ_i , and so τ_i must be a \mathcal{D} -bimodule map. By Meyer's theorem we may extend each τ_i to a unital completely positive map τ_i^1 on \mathcal{B}_i^1 . By Boca's theorem there exists a unital \mathcal{D} -bimodule completely positive map $\tau^1 := *\tau_i^1 : *\mathcal{B}_i^1 \rightarrow B(H)$ such that $\tau^1 \circ \iota_i = \tau_i^1$ and such that for $b_1 \in \ker E_{i_1}^1 = \ker E_{i_1}, \dots, b_n \in \ker E_{i_n}^1 = \ker E_{i_n}$ with $i_1 \neq i_2 \neq \dots \neq i_n$ one has

$$\tau(b_1 * \dots * b_n) = \tau_{i_1}(b_1) \dots \tau_{i_n}(b_n).$$

Hence, $\tau = \tau^1|_{*\mathcal{B}_i}$ is a joint completely contractive completely positive extension of τ_i with the same multiplicative property as above. Now, since each \mathcal{A}_i is generated as a \mathcal{D} -bimodule by $\mathcal{A}_i^0 := \mathcal{A}_i \cap \ker E_i$ and \mathcal{D} , every monomial in $\{\mathcal{A}_i\}_{i \in I}$ can always be written as a polynomial in $\{\mathcal{A}_i^0\}_{i \in I}$ with coefficients in \mathcal{D} , so that τ and π must coincide on polynomials in $\{\mathcal{A}_i\}_{i \in I}$. Hence, we see that $\tau|_{\mathcal{A}} = \pi|_{\mathcal{A}}$. Since $\pi|_{\mathcal{A}}$ has

the unique extension property we have that $\tau = \pi$, so that $\tau_i = \tau \circ \iota_i = \pi \circ \iota_i = \pi_i$. \square

5. FREE PRODUCTS OF GRAPH ALGEBRAS

In this section we will write $G = (V, E)$ for a directed graph, where the source and range maps are understood implicitly. Let $G = (V, E)$ be a directed graph. A function $c : E \rightarrow I$ is an I -coloring of edges of G , and we define an I -colored graph to be the triple (V, E, c) . Given such a colored directed graph (V, E, c) , we denote $E_i = c^{-1}(i)$.

Let $G = (V, E, c)$ be an I -colored directed graph and H a Hilbert space. A *Toeplitz-Cuntz-Krieger G -family* on H is a pair (P, S) comprised of a operators $P := \{P_v : v \in V\}$ on H and an I -tuple of sets of operators $S := \{S^{(i)}\}_{i=1}^d$ on H with $S^{(i)} := \{S_e^{(i)} : e \in c^{-1}(i)\}$ such that each $(P, S^{(i)})$ is a TCK family for (V, E_i) for each $i \in I$.

We say that (P, S) is a *Cuntz-Krieger G -family / full Cuntz-Krieger G -family* if, in addition, each $(P, S^{(i)})$ is a CK / full CK family for (V, E_i) for each $i \in I$ respectively.

From here on out, for a given set of vertices V , we set $\mathcal{V} := C_0(V)$. We will identify \mathcal{V} with $C^*(\{P_v\})$ for some (all) TCK or CK G -families (P, S) where $P_v \neq 0$ for all $v \in V$ and any colored graph G with V as its set of vertices.

Given a colored directed graph $G = (V, E, c)$, let $G_i = (V, c^{-1}(i))$ be the graph of color $i \in I$. By compounding universal properties, it is easy to see that TCK G -families are in bijection with $*$ -representations of the free product $\ast_{\mathcal{V}} \mathcal{T}(G_i)$ over $i \in I$ and that CK G -families are in bijection with $*$ -representations of the free product $\ast_{\mathcal{V}} \mathcal{O}(G_i)$ over $i \in I$.

As we saw in item (2) of Proposition 4.4, the existence of conditional expectations to the common subalgebra is desirable, so as to apply results that require the use of Boca's theorem. In [1], for the purpose of defining certain reduced free products of graph C^* -algebras, it was shown that *faithful* conditional expectations exist from $\mathcal{O}(G) \rightarrow \mathcal{V}$ when G is row-finite. When G is not necessarily row-finite, we can build conditional expectations on the level of the *Toeplitz* algebras $\mathcal{T}(G)$ instead.

Using the left regular representation $\pi_\ell : \mathcal{T}(G) \rightarrow B(H_G)$ given via the TCK family (P, S) as in the beginning of Section 3, we may define for each vertex $v \in V$ a norm one positive functional φ_v on $\mathcal{T}(G)$ by $\varphi_v(a) = \langle \pi_\ell(a) \xi_v, \xi_v \rangle$. Hence, we may define a contractive map

$\Psi_V : \mathcal{T}(G) \rightarrow B(H_G)$ by way of

$$\Psi_V(a) = \text{SOT-} \sum_{v \in V} \varphi_v(a) \cdot P_v.$$

Next, since each $a \in \mathcal{T}(G)$ is a norm limit of polynomials in $\{p_v\}_{v \in V}$ and $\{s_e\}_{e \in E}$, where (p, s) is a TCK family generating $\mathcal{T}(G)$, and since for each monomial $m \in \mathcal{T}(G)$ we have $\varphi_v(m)$ is non-zero for at most one vertex, we see the SOT-sum above is in fact a norm-convergent-sum for every $a \in \mathcal{T}(G)$. Hence the range of Ψ_V is the C^* -algebra $C^*(P)$ generated by $\{P_v\}_{v \in V}$, which is isomorphic to $C^*(p) \subseteq \mathcal{T}(G)$ via the isomorphism θ mapping P_v to p_v . Hence, the composition $\Phi_V = \theta \circ \Psi_V$ is a contractive idempotent. By a theorem of Tomiyama [11, Theorem 1.5.10], we have that $\Phi_V : \mathcal{T}(G) \rightarrow \mathcal{T}(G)$ is a conditional expectation onto $C^*(p) \cong \mathcal{V}$.

We next characterize those representations of the free product of Toeplitz graph algebras with the unique extension property when restricted to the free product of graph tensor algebras. Recall that for countable directed graphs $\{G_i\}_{i \in I}$ on the same vertex set V , by Proposition 4.3, we have that the embedding of ${}_{\mathcal{V}}^* \mathcal{T}_+(G_i)$ in ${}_{\mathcal{V}}^* \mathcal{T}(G_i)$ is completely isometric, so we may identify the former as a subalgebra of the latter.

Proposition 5.1. *Let $\{G_i\}_{i \in I}$ be a collection of countable directed graphs on the same vertex set V , and let $\pi : {}_{\mathcal{V}}^* \mathcal{T}(G_i) \rightarrow B(H)$ be a $*$ -representation. Then $\pi|_{{}_{\mathcal{V}}^* \mathcal{T}_+(G_i)}$ has the unique extension property if and only if for each $i \in I$ the $*$ -representation $\pi_i := \pi|_{\mathcal{T}(G_i)}$ is full CK with respect to G_i .*

Proof. The above constructed conditional expectation satisfies the conditions of item (2) in Proposition 4.4. Hence, $\pi : {}_{\mathcal{V}}^* \mathcal{T}(G_i) \rightarrow B(H)$ has the unique extension property if and only if each π_i has the unique extension property. Thus, by Theorem 3.5, this occurs if and only if each π_i is a full CK with respect to G_i . \square

We apply Proposition 5.1 to draw a dilation result that generalizes Corollary 3.7.

Corollary 5.2. *Let $G = (V, E, c)$ be an I -colored directed graph, and let (P, S) be a TCK G -family on H . Then there exists a full CK G -family (Q, T) on a Hilbert space K containing H , such that $f(P, S) = P_H f(Q, T)|_H$ for any polynomial $f \in \mathbb{C}\langle V, E \rangle$ in non-commuting variables.*

Proof. Let $\pi_{P,S} : {}^*\mathcal{T}(G_i) \rightarrow B(H)$ be the $*$ -representation associated to (P, S) . By [16, Theorem 1.2] we can dilate $\pi_{P,S}|_{{}^*\mathcal{T}_+(G_i)}$ to a maximal completely contractive homomorphism $\tau : {}^*\mathcal{T}_+(G_i) \rightarrow B(K)$. Without loss of generality, H is a subspace of K . Let $\rho : {}^*\mathcal{T}(G_i) \rightarrow B(K)$ be its unique extension to a $*$ -representation, and (Q, T) the associated TCK family of ρ . As τ has the unique extension property, by Proposition 5.1 we must have that (Q, T) is full CK, and as τ dilates $\pi_{P,S}|_{{}^*\mathcal{T}_+(G_i)}$, we have that every polynomial $f \in \mathbb{C}\langle V, E \rangle$ in non-commuting variables must satisfy $f(P, S) = P_H f(Q, T)|_H$. \square

The following result mirrors [37, Proposition 1.6] on the existence of maximal fully co-isometric summands, but when restricting to the context of directed graphs our result is more general as it requires no relations between families of different color.

Corollary 5.3. *Let $G = (V, E, c)$ be an I -colored directed graph, and let (P, S) be a TCK G -family on H . Then there is a unique maximal common reducing subspace K for operators in (P, S) such that $(P, S)|_K$ is full CK.*

Proof. Let $\pi_{P,S}$ be the $*$ -representation associated to (P, S) . By Proposition 2.3 there is a unique largest reducing subspace K for $\pi_{P,S}$ such that $\rho : {}^*\mathcal{T}(G_i) \rightarrow B(K)$ given by $\rho(b) = \pi_{P,S}(b)|_K$ has the unique extension property when restricted to ${}^*\mathcal{T}_+(G_i)$. By Proposition 5.1, we see that the associated TCK family $(P, S)|_K$ is in fact full CK, and K is a unique maximal common reducing subspace with this property. \square

Denote by $H_V = \bigoplus_{v \in V} H_v$ a Hilbert space direct sum of separable infinite dimensional Hilbert spaces H_v . Then $\mathcal{V} \cong C^*(\{p_v\})$ can be represented on H_V by mapping p_v to the projection P_v onto H_v , and if $\rho : \mathcal{T}(G_i) \rightarrow B(H)$ is a non-degenerate representation such that $\rho|_{C^*(\{p_v\})} : \mathcal{V} \cong C^*(\{p_v\}) \rightarrow B(H)$ where $\rho(p_v)H$ is infinite dimensional for each $v \in V$, then ρ is unitarily equivalent to a representation on H_V where $\rho(p_v) = P_v$ is the projection onto H_v .

When all G_i are row-finite, Duncan [17, Proposition 4.4] explained how ${}^*\mathcal{T}_+(G_i)$ has the unique extension property in ${}^*\mathcal{O}(G_i)$. We next prove this while providing the converse.

Theorem 5.4. *Let $\{G_i\}_{i \in I}$ be a collection of countable directed graphs over the same vertex set V . Then the quotient map $q : {}^*\mathcal{T}(G_i) \rightarrow$*

${}^*\mathcal{O}(G_i)$ is completely isometric on ${}^*\mathcal{T}_+(G_i)$. Hence $C_e^*({}^*\mathcal{T}_+(G_i))$ is a quotient of ${}^*\mathcal{O}(G_i)$.

Moreover, we have that each G_i is row-finite if and only if ${}^*\mathcal{T}_+(G_i)$ has the unique extension property in ${}^*\mathcal{O}(G_i)$. In particular, in this case we have $C_e^*({}^*\mathcal{T}_+(G_i)) \cong {}^*\mathcal{O}(G_i)$.

Proof. As $\mathcal{T}_+(G_i)$ can be identified as a subalgebra of $\mathcal{O}(G_i)$ via the image of the quotient map $q_i : \mathcal{T}(G_i) \rightarrow \mathcal{O}(G_i)$, by Proposition 4.3, we see that ${}^*\mathcal{T}_+(G_i)$ can be identified as a subalgebra of ${}^*\mathcal{O}(G_i)$ via the image of $q = {}^*q_i$.

For the second part, suppose each G_i is row-finite. Let $\pi : {}^*\mathcal{O}(G_i) \rightarrow B(H)$ be a $*$ -representation. Then $\pi \circ q$ is a $*$ -representation of ${}^*\mathcal{T}(G_i)$, and by invariance of the UEP it will suffice to show that $(\pi \circ q)|_{{}^*\mathcal{T}_+(G_i)}$ has the UEP. By Proposition 5.1, this happens if and only if $\pi_i \circ q_i$ is full CK. However, as each G_i is row-finite, Theorem 3.5 implies that each $\pi_i \circ q_i : \mathcal{T}(G_i) \rightarrow B(H)$ is full CK. Hence, ${}^*\mathcal{T}_+(G_i)$ has the unique extension property in ${}^*\mathcal{O}(G_i)$.

For the converse, we will construct a $*$ -representation of ${}^*\mathcal{O}(G_i)$ that lacks the unique extension property when restricted to ${}^*\mathcal{T}_+(G_i)$. Indeed, as one of $\{G_i\}$ is not row-finite, by Theorem 3.9 there is some $j \in I$ for which there is a CK representation $\rho_j : \mathcal{T}(G_j) \rightarrow B(H)$ that is not a full CK representation, and up to inflating H we may assume $\rho_j = \rho_j^{(\infty)}$. For $i \in I$ different from j , let $\rho_i : \mathcal{T}(G_i) \rightarrow B(H)$ be any representation annihilating the Cuntz-Krieger ideal $\mathcal{J}(G_i)$ for which $\rho_i(p_v) \neq 0$ for all $v \in V$. Again up to inflating H we may assume $\rho_i = \rho_i^{(\infty)}$. In this case for all $i \in I$ the representation ρ_i is unitarily equivalent to a representation on H_V where $\rho_i(p_v)$ is mapped to the projection P_v . In this case the free product ${}^*\rho_i$ is well-defined, and by Proposition 5.1 ${}^*\rho_i$ does not have the unique extension property when restricted to ${}^*\mathcal{T}_+(G_i)$ while still annihilating $\mathcal{J} = \langle \mathcal{J}(G_i) \rangle_{i \in I}$, so that it induces a representation ${}^*\dot{\rho}_i$ on ${}^*\mathcal{O}(G_i)$ that does not have the unique extension property. \square

Remark 5.5. After the completion of this work we were informed by Evgenios Kakariadis, that it is possible to show that $C_e^*({}^*\mathcal{T}_+(G_i)) \cong$

$*\mathcal{O}_v(G_i)$, regardless of whether the graphs G_i are row-finite or not. This completes the picture described in Theorem 5.4.

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